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An extended $U(3)_L \bigotimes U(3)_R$ chiral effective field theory which includes pseudoscalar and vector meson nonets as dynamic variables is presented. The theory combines a hidden symmetry approach with a general procedure of including the η' meson into chiral theory, and accounts for *direct* and *indirect* symmetry breaking effects via a mechanism based on the quark mass matrix. The theory is applied to anomalous radiative decays using particle mixing schemes, corresponding to different symmetry breaking assumptions and uniquely determined by the lagarngian presumed. Radiative decays of light flavor mesons are best explained within the framework of a one mixing angle scheme and provide evidence for $SU(3)_F$ and nonet symmetry breaking.

11.10.Ef; 11.30.Hv; 12.39.Fe; 13.40.Hq; 14.40.Aq.

I. INTRODUCTION

The decays of light flavor mesons have been discussed vigorously in the literature [1]- [25]. Particularly, phenomenological models based on effective field theories have been rather successful to explain anomalous and non-anomalous processes involving vector and pseudoscalar mesons, tensor and higher-spin mesons, J/ψ decays into a vector and a pseudoscalar meson, and many other related decays and topics [3,5,12,14,15]. Particularly successful are effective field theories based on the Hidden Local Symmetry (HLS) approach [26], which seems to provide an accurate and consistent framework to explain a vast amount of data, and more importantly, reflect on the energy properties of the more fundamental QCD lagrangian.

In the chiral limit, the QCD lagrangian exhibits an $SU(3)_L \bigotimes SU(3)_R$ symmetry which breaks down spontaneously to $SU(3)_V$, giving rise to an octet of pseudoscalar light Goldstone bosons. The QCD spectrum contains a ninth singlet boson because the axial $U(1)$ symmetry is broken by the anomaly. Nowadays it is well accepted that the lightest pseudoscalar mesons (π^+, π^-, π^0, K^+, K^-, K^0, $\bar{\mathrm{K}}^0$ and η) are candidates of the Goldstone octet. Though considerably heavier, the $\eta'(957\ MeV)$ meson is considered to be the pseudoscalar singlet.

Our main interest in the present work is to develop and apply a $U(3)_L \bigotimes U(3)_R$ chiral theory which includes the pseudoscalar and vector meson nonets as dynamic degrees of freedom. The theory combines the hidden local symmetry (HLS) approach of Bando et al. [26] with a general and universal procedure of including the η' meson into a chiral theory [27–30]. The $SU(3)_L \bigotimes SU(3)_R$ local symmetry based QCD lagrangian is extended by adding a term proportional to the topological charge operator (the so called winding number density), which gives rise to the well known Wess-Zumino-Witten (WZW) anomaly term and, renders the lagrangian to be $U(3)_L \bigotimes U(3)_R$ locally symmetric [27]. Most importantly, the lagrangian constructed, exhibits the fundamental symmetries of this extended QCD lagrangian and accounts for *direct* and *indirect* symmetry breaking effects via a mechanism based on the quark mass matrix. A natural way to account for $SU(3)$ flavor symmetry breaking terms involves an expansion of the QCD generating functional in powers of the quark masses. Here as well we use a similar expansion to generate symmetry breaking terms. We believe that the theory proposed defines an accurate framework for electro-weak and strong interactions of light flavor mesons and can be extended easily to include tensor and higher spin mesons as well as baryons.

Our paper is organized as follows. In section II we construct the lagrangian. In order to be fully consistent with QCD, a symmetry breaking companion is added to each of the unbroken lagrangian terms. These symmetry breaking companions affect the kinetic and mass terms of the lagrangian which, in turn determine the mixing schemes of state and coupling constants [31]. Two alternative ways are proposed for corresponding to $SU(3)_F$ symmetry breaking (*Alternative I*) and $U(3)_F$ symmetry breaking (*Alternative II*). The strong condition of nonet symmetry often

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used to relate the octet and singlet pseudoscalar states [2,6,17,8,10]is not presumed *a priori*. In the limit of nonet symmetry (in the sense of equal singlet and octet radiative decay constants, $F_0 = F_8$) the *Alternative II* scheme reduces to a scheme equivalent to that of the quark flavor basis (QFB) of Feldmann et al. [10–12]. The realization of these mixing schemes is described in section III. In section IV we apply our model to study anomalous processes. Namely, we calculate radiative decay widths of $V^0 \to P^0\gamma$, $P^0 \to V^0\gamma$ and $P^0 \to \gamma\gamma$, with $P^0 = \pi, K, \eta, \eta'$ and $V^0 = \rho, K^*, \omega, \phi$. Global fit to data is performed in order to determine numerically the symmetry breaking scales and evaluate the success of the different schemes to explain data. We summarize and conclude in section V.

II. THE EFFECTIVE LAGRANGIAN

The main objective in the present section is to construct an effective lagrangian, $U(3)_L \bigotimes U(3)_R$ local symmetry based, for pseudoscalar and vector meson nonets interacting with external electroweak fields. As a non-linear representation of the pseudoscalar nonet fields we define [27,28],

$$U(P, \eta_0 + F_0\vartheta) \equiv \xi^2(P, \eta_0 + F_0\vartheta) \equiv \exp\left\{ i\frac{\sqrt{2}}{F_8}P + i\sqrt{\frac{2}{3}}\frac{1}{F_0}(\eta_0 + F_0\vartheta)\mathbf{1} \right\} , \tag{2.1}$$

where $\eta_0(x)$ stands for the pseudoscalar singlet and $P(x)$ for the pseudoscalar Goldstone octet matrix,

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix} , \tag{2.2}$$

with obvious notation. In this representation, the octet degrees of freedom are contained in the unimodular part of the field U while the phase $detU = \exp i\left\{i\sqrt{6}(\eta_0/F_0 + \vartheta)\right\}$ involves the singlet only. The vacuum angle serves here as an auxiliary field $\vartheta(x)$ which renders $detU$ to be invariant under $U(3)_L \bigotimes U(3)_R$ transformations [27,28]. Since there exists no dimension-nine representation for $U(3)_L \bigotimes U(3)_R$, the octet (F_8) and singlet (F_0) radiative decay constants may well be different. We follow Refs. [32–34] and define vector type Γ_μ and axial-vector type Δ_μ covariants,

$$\Gamma_\mu = \frac{i}{2}\left[\xi^\dagger, \partial_\mu\xi\right] + \frac{1}{2}\left(\xi^\dagger r_\mu \xi + \xi l_\mu \xi^\dagger\right) , \tag{2.3}$$

$$\Delta_\mu = \frac{i}{2}\left\{\xi^\dagger, \partial_\mu\xi\right\} + \frac{1}{2}\left(\xi^\dagger r_\mu \xi - \xi l_\mu \xi^\dagger\right) . \tag{2.4}$$

Here r_μ and l_μ represent the standard model external gauge fields; $r_\mu = v_\mu + a_\mu$ and $l_\mu = v_\mu - a_\mu$, with v_μ and a_μ being the vector and axial vector external electroweak fields, respectively. For pure electromagnetic interactions $l_\mu = r_\mu = -eQA_\mu$ where A_μ denotes the electromagnetic field and $Q = diag(2/3, -1/3, -1/3)$ the quark charge operator.

Under $U(3)_L \bigotimes U(3)_R$ the field, Eqn. 2.1, transforms as,

$$U' = RUL^\dagger, \tag{2.5}$$

with $R \in U(3)_R$, $L \in U(3)_L$. The vector (Γ_μ) and axial-vector (Δ_μ) covariants transform, respectively, as a gauge and matter fields, i.e.,

$$\Gamma'_\mu = K\Gamma_\mu K^\dagger + iK\partial_\mu K^\dagger , \tag{2.6}$$

$$\Delta'_\mu = K\Delta_\mu K^\dagger , \tag{2.7}$$

where $K(U, R, L)$ is a compensatory field representing an element of the conserved vector subgroup $U(3)_V$ [32–34].

The dynamical gauge bosons are defined as a 3×3 vector field matrix V_μ which transforms as,

$$V'_\mu = KV_\mu K^\dagger + \frac{i}{g}K\partial_\mu K^\dagger . \tag{2.8}$$

Clearly, the vector $\Gamma_\mu - gV_\mu$ as well as the axial vector Δ_μ transform homogeneously. Thus to lowest order (i.e. with the smallest number of derivatives) the lagrangian can be constructed from the traces of Δ_μ^2 , $(\Gamma_\mu - gV_\mu)^2$,

Δ_μ, $(\Gamma_\mu - gV_\mu)$, $D^\mu\vartheta$, and arbitrary functions of the variable $X(x) \equiv \sqrt{6}\eta_0(x)/F_0 + \vartheta(x)$ all of which being invariant under $U(3)_L \bigotimes U(3)_R$ transformations. Then to lowest order, a most general form of a symmetric effective chiral lagrangian is,

$$L = L_A + aL_V - \frac{1}{2}Tr(V_{\mu\nu}V^{\mu\nu}) \ , \tag{2.9}$$

with,

$$\begin{aligned} L_A &= W_1(X)Tr(\Delta_\mu\Delta^\mu) + W_4(X)Tr(\Delta_\mu)Tr(\Delta^\mu) + \\ &W_5(X)Tr(\Delta_\mu)D^\mu\vartheta + W_6(X)D_\mu\vartheta D^\mu\vartheta \ , \end{aligned} \tag{2.10}$$

$$\begin{aligned} L_V &= \tilde{W}_1(X)Tr([\Gamma_\mu - gV_\mu][\Gamma^\mu - gV^\mu]) + \\ &\tilde{W}_4(X)Tr(\Gamma_\mu - gV_\mu)Tr(\Gamma^\mu - gV^\mu) \ , \end{aligned} \tag{2.11}$$

and,

$$D_\mu\vartheta = \partial_\mu\vartheta + Tr(r_\mu - l_\mu) \ , \tag{2.12}$$

$$V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - ig[V_\mu, V_\nu] \ . \tag{2.13}$$

All three terms of the lagrangian L in Eqns.2.9-2.11 are invariant under $U(3)_L \bigotimes U(3)_R$ transformations. The coefficient functions, W_i, and $\tilde{W}_i$, are constrained by parity conservation to be even functions of the variable X. In addition by requiring that the normalization of the HLS symmetric kinetic term be equal 1/2, it is easy to show that $W_1(0) = F_8^2$, $W_4(0) = (F_0^2 - F_8^2)/3$, and $W_6(0) = 1/2$. Such a normalization ensures that the pseudoscalar singlet couples to the singlet axial current with a strength F_0 while the octet states couple to the octet axial currents with a strength F_8. Although the lagrangian, Eqn.2.9, appears similar to that of Bando et al. [26], the terms L_A and L_V are different. Namely, including the η' meson as a dynamical variable involves additional terms with $Tr(\Delta_\mu)Tr(\Delta^\mu)$, $Tr(\Delta_\mu)D^\mu\vartheta$ and $Tr(\Gamma^\mu - gV^\mu)Tr(\Gamma_\mu - gV_\mu)$. In addition we have introduced the coefficient functions $W_i(X)$ which are absent in the $SU(3)$ symmetry limit. The kinetic term of the pseudoscalar mesons as well as their strong and electroweak interactions with the Goldstone fields are all included in L_A. As for the vector mesons, L_V incorporates all interactions with the pseudoscalar fields. The kinetic term is written explicitly as $\frac{1}{2}Tr(V_{\mu\nu}V^{\mu\nu})$. Similar to the lagrangian of Bando et al. [26], however, the sum $L_A + aL_V$ contains, amongst other contributions, a vector meson mass term $\sim V_\mu V^\mu$, a vector-photon conversion factor $\sim V_\mu A^\mu$ and the coupling of pseudoscalar pairs to both vectors and photons. The latter coupling can be eliminated by choosing a value $a = 2$, which allows incorporating the conventional vector-dominance in the electromagnetic form-factors of pseudoscalar mesons [26], and eliminating the coupling of a pseudoscalar particle to two photons. Some recent data analysis [13] with the model lagrangian of Bando et al. [26] argue for a value $a = 2.4$. In view of the new terms added in our lagrangain, this parameter must be studied anew.

A. $SU(3)_F$ symmetry breaking

As already indicated in the introduction, the expansion of the QCD generating functional in powers of quark mass term results with SU(3) flavor symmetry breaking terms. There is no unique way to introduce such terms into the lagrangian, but it is quite natural choosing them to be (i) Hermitian, (ii) proportional to powers of the quark mass matrix and, (iii) recover the unbroken lagrangian smoothly in the limit of vanishing symmetry breaking parameters. We stress that in order to be consistent with QCD one must add a symmetry breaking companion to each term of the unbroken lagrangian. This can be accomplished following the procedure described below. First, the Goldstone meson mass degeneracy is removed by adding a $U(3)_L \bigotimes U(3)_R$ symmetry violating mass term. Most generally such a term reads, [27,28,30],

$$L_m = -W_0(X) + W_2(X)Tr\chi_+ + iW_3(X)Tr\chi_- \ , \tag{2.14}$$

with,

$$\chi_\pm = 2B_0(\xi\mathcal{M}^\dagger\xi \pm \xi^\dagger\mathcal{M}\xi^\dagger) \ , \qquad B_0 = m_\pi^2/(m_u + m_d) \ , \tag{2.15}$$

where $\mathcal{M} = diag(m_u, m_d, m_s)$. Next, in order to incorporate symmetry breaking corresponding to each of the unbroken L_A and L_V lagrangian densities, we define a universal Hermitian matrix B,

$$B \equiv \frac{1}{4B_0(m_u+m_d+m_s)}\chi_+ \ . \tag{2.16}$$

Then, symmetry breaking companions for L_A are constructed in two alternative ways. The first (hereafter referred to as *Alternative I*) breaks octet symmetry ($SU(3)_F$) only, and corresponds to a quadratic form of the Goldstone meson kinetic energy term. Let,

$$U_8 \equiv \xi_8^2 \equiv \exp(i\frac{\sqrt{2}}{F_8}P) \tag{2.17}$$

be the pure octet field matrix and let,

$$\bar{\Delta}_\mu = \frac{i}{2}\left\{\xi_8^\dagger, \partial_\mu \xi_8\right\} + \frac{1}{2}\left(\xi_8^\dagger r_\mu \xi_8 - \xi_8 l_\mu \xi_8^\dagger\right) \ , \tag{2.18}$$

be the octet axial vector covariant. Then a general symmetry breaking lagrangian $\bar{L}_A$ would be,

$$\begin{aligned}\bar{L}_A = W_1(X)\left(c_A Tr(B\bar{\Delta}_\mu\bar{\Delta}^\mu) + d_A Tr(B\bar{\Delta}_\mu B\bar{\Delta}^\mu)\right) + \\ W_4(X) d_A Tr(B\bar{\Delta}_\mu) Tr(B\bar{\Delta}^\mu) + W_5(X) c_A Tr(B\bar{\Delta}_\mu) D^\mu\vartheta \ .\end{aligned} \tag{2.19}$$

The second alternative (*Alternative II*) breaks $U(3)_F$ symmetry and uses the nonet axial vector Δ^μ (instead of $\bar{\Delta}^\mu$ as above). Namely,

$$\begin{aligned}\bar{L}_A = W_1(X)\left(c_A Tr(B\Delta_\mu\Delta^\mu) + d_A Tr(B\Delta_\mu B\Delta^\mu)\right) \\ + W_4(X)\left(c_A Tr(B\Delta_\mu) Tr(\Delta^\mu) + d_A Tr(B\Delta_\mu) Tr(B\Delta^\mu)\right) + \\ W_5(X) c_A Tr(B\Delta_\mu) D^\mu\vartheta \ .\end{aligned} \tag{2.20}$$

This expression gives a bilinear meson kinetic energy term. Similarly the asymmetric companion of L_V would be,

$$\begin{aligned}\bar{L}_V = \tilde{W}_1(X)\left(c_V Tr(B[\Gamma_\mu - gV_\mu][\Gamma^\mu - gV^\mu]) + \right. \\ d_V Tr(B[\Gamma_\mu - gV_\mu] B[\Gamma^\mu - gV^\mu])) + \\ \tilde{W}_4(X)\left(c_V Tr(\Gamma_\mu - gV_\mu) Tr(B[\Gamma^\mu - gV^\mu])\right. \\ \left. + d_V Tr(B[\Gamma_\mu - gV_\mu]) Tr(B[\Gamma^\mu - gV^\mu])\right) \ .\end{aligned} \tag{2.21}$$

In the expressions above, c_A, d_A, c_V and d_V are the model symmetry breaking parameters to be determined from data analyses. It is to be stressed that $\bar{L}_A$ and $\bar{L}_V$ differ also from the ones defined by Bando et al. [26] and Bramon et al. [5]. First, like our symmetric L_A and L_V the asymmetric companions $\bar{L}_A$ and $\bar{L}_V$ involve additional terms which are absent in the $SU(3)$ limit. Secondly, the constants d_i of Bando et al. [26] are chosen arbitrarily to be $d_i = c_i^2$ and as will be demonstrated through detailed data analysis this may not be a well justified assumption. Thirdly, unlike Bando et al. [26] the symmetry breaking matrix B is Hermitian. It is similar (but not identical) to that of Bramon et al. [5] and rather close to the one proposed by Benayoun and O'Connell [14]. Furthermore the lagrangian is constructed in close analogy with QCD and allows for symmetry breaking in a universal manner. Particularly, the matrix B, Eqn. 2.16, enables us to maintain in our theory, the QCD ratios of isospin to $SU(3)$ symmetry breaking scales.

Summing all terms, including the well known Wess-Zumino-Witten term L_{WZW} [35,36], the asymmetric lagrangian assumes the form,

$$L = L_A + \bar{L}_A + L_m + a(L_V + \bar{L}_V) - \frac{1}{4}Tr(V_{\mu\nu}V^{\mu\nu}) + L_{WZW} + \ldots \ , \tag{2.22}$$

where "..." stands for terms accounting for the regularization of one loop contributions [3,4,27]. Note that $\bar{L}_V$ includes vector meson mass terms which also violate symmetry. Generally speaking, a symmetry breaking companion for the vector meson kinetic term $\frac{1}{2}Tr(V_{\mu\nu}V^{\mu\nu})$ should have been added also. However, at present, no evidence exists for such asymmetric terms and therefore will not be considered in the discussion to follow. The lagrangian constructed above should describe the mass splitting of the pseudoscalar and vector mesons (see for example [37]). Since the ground state of the field U is proportional to the unit matrix, one can set the auxiliary field $\vartheta = 0$ [27]. With these simplifying assumptions the quantities W_i and $\tilde{W}_i$ become functions of the singlet field (η_0) only. To lowest order their expansions read,

$$W_0 = const + F_8^4 \left(w_0 \frac{\eta_0^2}{F_0^2} + \dots \right) \ , \tag{2.23}$$

$$W_1 = F_8^2 \left(1 + w_1 \frac{\eta_0^2}{F_0^2} + \dots \right) \ , \tag{2.24}$$

$$W_2 = \frac{F_8^2}{4} \left(1 + w_2 \frac{\eta_0^2}{F_0^2} + \dots \right) \ , \tag{2.25}$$

$$W_3 = \frac{F_8^2}{2} (w_3 \frac{\eta_0}{F_0} + \dots) \ , \tag{2.26}$$

$$W_4 = \frac{F_0^2 - F_8^2}{3} \left(1 + w_4 \frac{\eta_0^2}{F_0^2} + \dots \right) \tag{2.27}$$

$$W_5 = F_8^2 \left(\bar{w}_5 + w_5 \frac{\eta_0^2}{F_0^2} + \dots \right) \tag{2.28}$$

$$W_6 = F_8^2 \left(\bar{w}_6 + w_6 \frac{\eta_0^2}{F_0^2} + \dots \right) \ , \tag{2.29}$$

$$\tilde{W}_1 = F_8^2 \left(1 + \tilde{w}_1 \frac{\eta_0^2}{F_0^2} + \dots \right) \ , \tag{2.30}$$

$$\tilde{W}_4 = F_8^2 \left(1 + \tilde{w}_4 \frac{\eta_0^2}{F_0^2} + \dots \right) \ . \tag{2.31}$$

Here the coefficients $w_i, \bar{w}_i$ and $\tilde{w}_i$ are free parameters not yet determined. In the applications to be discussed below, however, only a few combinations (rather than each) of these parameters are needed.

III. SYMMETRY BREAKING AND PSEUDOSCALAR MESON MIXING

Symmetry breaking terms affects the form of the kinetic and mass lagrangian densities of the Goldstone mesons which in turn uniquely determine the state and decay constants mixing schemes [31]. In what follows we realize the mixing schemes, corresponding to the *Alternative I* (Eqn. 2.19) and the *Alternative II* (Eqn. 2.20) lagrangians. These two alternatives allow us to evaluate $SU(3)_F$ *versus* $U(3)_F$ symmetry breaking. A third scheme corresponding to the QFB of Feldmann et al. [10], is also realized from the *Alternative II* by requiring nonet symmetry ($F_0 = F_8$).

Generally, for any EFT the kinetic and mass terms can be written in the form,

$$L_{km} = \frac{1}{2} (\partial_\mu \Phi) \mathcal{K} (\partial^\mu \Phi) + \frac{1}{2} \Phi \mathcal{M}^2 \Phi \ , \tag{3.1}$$

where Φ represents the intrinsic meson field matrix, $\mathcal{K}$ and $\mathcal{M}^2$ are the kinetic and mass matrices. In the presence of symmetry breaking $\mathcal{K}$ and $\mathcal{M}^2$ are non-diagonal, and Eqn. 3.1 does not have the standard quadratic form as invoked by the Klein-Gordon equation for the physical fields. It can be reduced into this standard form by applying three consecutive steps, which transform the intrinsic fields into the physical meson fields according to [31],

$$\Phi = \Theta \Phi_{ph} \ ; \quad \Theta = \Upsilon R \Omega \ . \tag{3.2}$$

Here Υ represents a unitary transformation which diagonalize the kinetic matrix K, R stands for rescaling of the fields, and Ω is another unitary matrix which diagonalize the resulting mass matrix. Let us work in details a general scheme for the mixing of two states, where presumably, the η and η' are linear combinations of an octet(η_8) and a singlet (η_0), Only. Using the symmetry breaking *Alternative II* lagrangian, Eqn.2.20, the kinetic and mass terms are bilinear and the corresponding matrices read,

$$\mathcal{K} = \begin{pmatrix} \kappa_\pi & 0 & 0 & 0 \\ 0 & \kappa_K & 0 & 0 \\ 0 & 0 & \kappa_{88} & \kappa_{08} \\ 0 & 0 & \kappa_{08} & \kappa_{00} \end{pmatrix} \ , \qquad \mathcal{M}^2 = \begin{pmatrix} \mu_\pi^2 & 0 & 0 & 0 \\ 0 & \mu_K^2 & 0 & 0 \\ 0 & 0 & \mu_{88}^2 & \mu_{08}^2 \\ 0 & 0 & \mu_{08}^2 & \mu_{00}^2 \end{pmatrix} \ . \tag{3.3}$$

Here,

$$
\begin{aligned}
\kappa_\pi &= 1 \ , \\
\kappa_K &= 1 + \frac{1}{2} c_A \ , \\
\kappa_{88} &= 1 + \frac{2}{3} \left(c_A + \frac{2r^2+1}{3r^2} d_A \right) \ , \\
\kappa_{08} &= -\frac{r\sqrt{2}}{6} \left(\frac{1+r^2}{r} c_A + 2\frac{2r^2+1}{3r} d_A \right) \ , \\
\kappa_{00} &= 1 + \frac{1+5r^2}{18} c_A + \frac{1+r^2}{6} d_A \ , \\
\mu_\pi^2 &= m_\pi^2 = 2mB_0 \ , \\
\mu_K^2 &= \frac{m_\pi^2}{2}(1 + \frac{m_s}{m}) \ , \\
\mu_{88}^2 &= \frac{m_\pi^2}{3}(1 + 2\frac{m_s}{m}) \ , \\
\mu_{08}^2 &= \frac{\sqrt{2}}{3} m_\pi^2 (1 - \frac{m_s}{m})(1 + \sqrt{6} w_3) r \ , \\
\mu_{00}^2 &= \frac{m_\pi^2}{3}(2 + \frac{m_s}{m}) \left[1 + 2\sqrt{6} w_3 - 3w_2 + \frac{6mF_8^2}{m_\pi^2(2m+m_s)} w_0 \right] r^2 \ ,
\end{aligned}
\tag{3.4}
$$

and $r = F_8/F_0$ is a measure of nonet symmetry breaking. For simplicity , small terms of order $\sim c_A m/m_s,\ d_A m/m_s$ are neglected and isospin symmetry $(m_u = m_d = m)$ is assumed in the expressions listed above. Note that $\mathcal{K}$ and $\mathcal{M}^2$ have non-diagonal 2×2 submatrices, and as demonstrated in Ref. [31] the transformation Θ involves two mixing angles and two rescaling parameters. First as in Ref. [31] we diagonalize $\mathcal{K}$ using the unitary transformation,

$$
\begin{pmatrix} \pi \\ K \\ \eta_8 \\ \eta_0 \end{pmatrix} == \Upsilon \begin{pmatrix} \pi \\ K \\ \bar\eta_8 \\ \bar\eta_0 \end{pmatrix} \ , \quad \Upsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\lambda & \sin\lambda \\ 0 & 0 & -\sin\lambda & \cos\lambda \end{pmatrix} \ . \tag{3.5}
$$

This leads to,

$$
L_{km} = (\partial_\mu \pi)^2 + \kappa_K (\partial_\mu K)^2 + \kappa_0 (\partial_\mu \bar\eta_0)^2 + (\bar\eta_8, \bar\eta_0) \Upsilon^{-1} \mathcal{M}^2 \Upsilon \begin{pmatrix} \bar\eta_8 \\ \bar\eta_0 \end{pmatrix} \ , \tag{3.6}
$$

with,

$$
\kappa_8 = \frac{1}{2} \left[\kappa_{88} + \kappa_{00} - \sqrt{(\kappa_{00} - \kappa_{88})^2 + 4\kappa_{80}^4} \right] \ , \tag{3.7}
$$

$$
\kappa_0 = \frac{1}{2} \left[\kappa_{88} + \kappa_{00} + \sqrt{(\kappa_{00} - \kappa_{88})^2 + 4\kappa_{80}^4} \right] \ , \tag{3.8}
$$

and,

$$
\frac{\tan\lambda}{1 - \tan^2\lambda} = \frac{\kappa_{80}}{\kappa_{00} - \kappa_{88}} \ . \tag{3.9}
$$

Next we rescale the fields $(\pi, K, \bar\eta_8, \bar\eta_0)$ into $(\pi, \hat K, \hat\eta_8, \hat\eta_0)$ using,

$$
R = diag(1\ z_K,\ z,\ f) = diag(1, \frac{1}{\sqrt{\kappa_K}}, \frac{1}{\sqrt{\kappa_8}}, \frac{1}{\sqrt{\kappa_0}}) \ . \tag{3.10}
$$

Following these two steps, the kinetic term acquires the standard quadratic form and the resulting mass matrix is,

$$
\tilde{\mathcal{M}}^2 = R \Upsilon^{-1} \mathcal{M}^2 \Upsilon R = \begin{pmatrix} \tilde\mu_\pi^2 & 0 & 0 & 0 \\ 0 & \tilde\mu_K^2 & 0 & 0 \\ 0 & 0 & \tilde\mu_{88}^2 & \tilde\mu_{80}^2 \\ 0 & 0 & \tilde\mu_{80}^2 & \tilde\mu_{00}^2 \end{pmatrix} \ , \tag{3.11}
$$

with,

$$\tilde{\mu}_\pi^2 = \mu_\pi^2 \ , \tag{3.12}$$
$$\tilde{\mu}_K^2 = \mu_K^2 z_K^2 \ , \tag{3.13}$$
$$\tilde{\mu}_{88}^2 = z^2 \left(\cos^2\lambda\mu_{88}^2 + \sin^2\lambda\mu_{00}^2 - 2\sin\lambda\cos\lambda\mu_{80}^2\right) \ , \tag{3.14}$$
$$\tilde{\mu}_{80}^2 = zf \left(\sin\lambda\cos\lambda(\mu_{88}^2 - \mu_{00}^2) + (\cos^2\lambda - \sin^2\lambda)\mu_{80}^2\right) \ , \tag{3.15}$$
$$\tilde{\mu}_{00}^2 = f^2 \left(\sin^2\lambda\mu_{88}^2 + \cos^2\lambda\mu_{00}^2 + 2\sin\lambda\cos\lambda\mu_{80}^2\right) \ . \tag{3.16}$$

As a last step we diagonalize the mass matrix, Eqn. 3.11, by applying a second unitary transformation,

$$\begin{pmatrix} \hat{\eta}_8 \\ \hat{\eta}_0 \end{pmatrix} = \Omega \begin{pmatrix} \eta \\ \eta' \end{pmatrix} \ , \quad \Omega = \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \ , \tag{3.17}$$

and take the eigenvalues to be equal to the masses of the physical particles, i.e.,

$$m_\pi^2 = \mu_\pi^2 \ , \tag{3.18}$$
$$m_K^2 = \mu_K^2 z_K^2 \ , \tag{3.19}$$
$$m_\eta^2 = \frac{1}{2}\left(\tilde{\mu}_{88}^2 + \tilde{\mu}_{00}^2 - \sqrt{(\tilde{\mu}_{88}^2 - \tilde{\mu}_{00}^2)^2 + 4(\tilde{\mu}_{80}^2)^2}\right) \ , \tag{3.20}$$
$$m_{\eta'}^2 = \frac{1}{2}\left(\tilde{\mu}_{88}^2 + \tilde{\mu}_{00}^2 + \sqrt{(\tilde{\mu}_{88}^2 - \tilde{\mu}_{00}^2)^2 + 4(\tilde{\mu}_{80}^2)^2}\right) \ , \tag{3.21}$$

and,

$$\frac{\tan\chi}{1-\tan^2\chi} = -\frac{\tilde{\mu}_{80}^2}{\tilde{\mu}_{88}^2 - \tilde{\mu}_{00}^2} \ . \tag{3.22}$$

By definition the physical η and η' fields are orthogonal and are related to the intrinsic fields via,

$$\begin{pmatrix} \eta_8 \\ \eta_0 \end{pmatrix} = \Theta' \begin{pmatrix} \eta \\ \eta' \end{pmatrix} \ , \tag{3.23}$$

where,

$$\Theta' = \begin{pmatrix} z\cos\lambda\cos\chi - f\sin\lambda\sin\chi & z\cos\lambda\sin\chi + f\sin\lambda\cos\chi \\ -z\sin\lambda\cos\chi - f\cos\lambda\sin\chi & -z\sin\lambda\sin\chi + f\cos\lambda\cos\chi \end{pmatrix} \ . \tag{3.24}$$

Using this transformation, we may rewrite the nonlinear representation, Eqn. 2.1 in terms of the physical fields, e.g.,

$$U = \exp i\frac{\sqrt{2}}{F_\pi}\mathcal{P} \ , \tag{3.25}$$

where $\mathcal{P}$ stands for the pseudoscalar nonet matrix,

$$\mathcal{P} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{1}{\sqrt{6}}(X_\eta\eta + X_{\eta'}\eta') & \pi^+ & z_s K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{1}{\sqrt{6}}(X_\eta\eta + X_{\eta'}\eta') & z_s K^0 \\ z_s K^- & z_s \bar{K}^0 & \frac{1}{\sqrt{6}}(Y_\eta\eta + Y_{\eta'}\eta') \end{pmatrix} \ . \tag{3.26}$$

The coefficients X_i and $Y_i, (i = \eta, \eta')$ are listed in Table I. It is worthy of mention that by taking the eigenvalues of the resulting mass matrix to be equal to the masses of the physical mesons we have eliminated the dependence on the model free parameters w_i; the mixing angles λ and χ, the rescaling parameters z and f, the coefficients X_i and Y_i, and the pseudoscalar nonet matrix $\mathcal{P}$ become functions of just three symmetry breaking scales c_A, d_A and r. Thus enable the pseudoscalar nonet matrix, Eqn. 3.26 incorporate symmetry breaking effects indirectly. We refer to these as *indirect* as opposed to *direct* symmetry breaking due to the broken $\bar{L}_A$ and $\bar{L}_V$ lagrangian companions.

At this stage some comments are in order :
(i) A better known expression of the transformation Θ is [9,18],

$$\Theta = \begin{pmatrix} \cos\theta_{\eta'} & \sin\theta_\eta \\ -\sin\theta_{\eta'} & \cos\theta_\eta \end{pmatrix} diag(z_1, z_2) \ , \tag{3.27}$$

Where,

$$z_1^2 = z^2\cos^2\chi + f^2\sin^2\chi, \quad z\cos\chi = z_1\cos\psi_1, \quad f\sin\chi = -z_1\sin\psi_1, \tag{3.28}$$
$$z_2^2 = f^2\cos^2\chi + z^2\sin^2\chi, \quad f\cos\chi = z_2\cos\psi_2, \quad z\sin\chi = -z_2\sin\psi_2, \tag{3.29}$$
$$\tan\psi_1 = -\frac{f}{z}\tan\chi, \qquad \tan\psi_1 = \frac{f^2}{z^2}\tan\psi_2, \tag{3.30}$$
$$\theta_\eta = \lambda - \psi_2 \ , \qquad \theta_{\eta'} = \lambda - \psi_1 \ . \tag{3.31}$$

This exact general expression of the field transformation involves two mixing angles θ_η and $\theta_{\eta'}$ as well as two field rescaling parameters z_1 and z_2. A similar two angle scheme was suggested by Escribano and Frère [18], however their transformation does not account for the field rescaling and violates orthogonality of the η and η' states.

(ii) With the symmetry breaking lagrangian $\bar{L}_A$, 2.19, the kinetic term assumes a quadratic form, with $\kappa_{88} = 1 + 2(c_A + d_A)/3$, $\kappa_{08} = 0$, and $\kappa_{00} = 1$. All other matrix elements of $\mathcal{K}$ and $\mathcal{M}^2$ remain the same as in Eqn. 3.4. Consequently, $z = 1/\sqrt{1 + 2(c_A + d_A)/3}$ and $f = 1$, and the angle λ vanishes, so that the *Alternative I* scheme involves a single mixing angle.

(iii) In the limit of exact nonet symmetry, $r = 1$ ($F_8 = F_0$), the pseudoscalar meson kinetic matrix is diagonal in the so called quark flavor basis (QFB), where the flavorless mesons are represented by fields with or without strange quark content ($\eta_s \sim s\bar{s}$ and $\eta_q \sim (u\bar{u} + d\bar{d})/\sqrt{2}$). With this basis the kinetic matrix is diagonal, what leads to a one mixing angle scheme. We stress that Feldmann et al. [10–12] do not account for field rescaling and therefore, their mixing angle ϕ is equivalent to our mixing angle which defines the transformation of the rescaled fields into the physical fields.

To conclude this section, we briefly consider the vector meson mixing problem. Here, the kinetic energy term has the standard quadratic form. Thus one needs diagonalizing the mass matrix only, which effectively, depends on four parameters $m_V^2 = 2F_8^2 ag$, c_V, d_V and w_4, where m_V stands for the vector meson nonet symmetric mass. All four parameters can be fixed by taking the calculated physical meson masses to be equal to their experimental values. Following a similar orthogonalization procedure as above, it is straightforward to show that the physical vector nonet matrix with non-ideal mixing reads,

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}}(\rho^0 + \omega + \epsilon\phi) & \rho^+ & K^{*+} \\ \rho^- & \frac{1}{\sqrt{2}}(-\rho^0 + \omega + \epsilon\phi) & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi - \epsilon\omega \end{pmatrix} \ . \tag{3.32}$$

with $\epsilon = 0.048$, being a measure of the non-strange (strange) admixture in ϕ (ω). Nearly the same value of ϵ is deduced from the width of the $\phi \to \pi\gamma$ decay (see next section). In practice, the vector meson mixing angle turns to be very close to the one of ideal mixing, and likewise, the corresponding strange (non-strange) admixtures in the nonet vector matrix are small. Since we worked in the exact isospin symmetry limit, the vector field, Eqn. 3.32, does not account for $\rho - \omega$ mixing. The derivation of the broken matrix is straightforward and will not be given here.

IV. RADIATIVE DECAY WIDTHS

The main objective in the following section is to apply our theory to calculate decay width of the anomalous processes listed in table IV. Details of the calculations are described in previous publications and will not be repeated (see for example Ref. [15]). Our analyses generalize the treatments of Refs. [3,5,13,15], in the sense that we treat all nonet mesons on equal footing and most importantly we use mixing schemes which are uniquely determined by the lagrangains presumed.

In our analyses we use pseudoscalar and vector field matrices expressed in terms of physical fields and therefore allow for *indirect* and *direct* symmetry breaking effects. The lagrangian is factorized in the form,

$$L_{P\gamma\gamma} = L_{P\gamma\gamma} + c_W \bar{L}_{P\gamma\gamma}, \tag{4.1}$$
$$L_{VP\gamma} = L_{VP\gamma} + c_W \bar{L}_{VP\gamma} \ , \tag{4.2}$$

where $L_{P\gamma\gamma}$ ($L_{VP\gamma}$) represents the overall contribution of the unbroken anomalous lagranagian to the $P\gamma\gamma$ ($VP\gamma$) interaction, and $\bar{L}_{P\gamma\gamma}$ ($\bar{L}_{VP\gamma}$) is the corresponding direct symmetry breaking companion. Here c_W stands for the direct symmetry breaking parameter. As mentioned above, both $L_{P\gamma\gamma}$ and L_{PVV} account for symmetry breaking via the pseudscalar and vector meson matrices. To write these terms explicitly, we start from the anomalous lagrangian [3],

$$L_{anomalous} = L^{(0)}_{VVP} + L_{WZW}(P\gamma\gamma) \ . \tag{4.3}$$

In terms of the covariants Δ_μ, $\Gamma_\mu - gV_\mu$, $V_{\mu\nu}$ and $\Gamma_{\mu\nu} = \partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu - i[\Gamma_\mu, \Gamma_\nu]$ (see sect II for definitions), the pseudoscalar-vector-vector (PVV) coupling, $L^{(0)}_{VVP}$, has at most six contributions,

$$\begin{aligned}
L^{(0)}_{VVP} = {} & g_1\epsilon^{\mu\nu\alpha\beta}Tr(V_{\mu\nu}[V_\alpha - \frac{1}{g}\Gamma_\alpha]\Delta_\beta) + g_2\epsilon^{\mu\nu\alpha\beta}Tr(\Gamma_{\mu\nu}[V_\alpha - \frac{1}{g}\Gamma_\alpha]\Delta_\beta) + \\
& g_3\epsilon^{\mu\nu\alpha\beta}Tr(V_{\mu\nu})Tr([V_\alpha - \frac{1}{g}\Gamma_\alpha]\Delta_\beta) + g_4\epsilon^{\mu\nu\alpha\beta}Tr(\Gamma_{\mu\nu})Tr([V_\alpha - \frac{1}{g}\Gamma_\alpha]\Delta_\beta) + \\
& g_5\epsilon^{\mu\nu\alpha\beta}Tr(V_{\mu\nu})Tr(V_\alpha - \frac{1}{g}\Gamma_\alpha)Tr(\Delta_\beta) + \\
& g_6\epsilon^{\mu\nu\alpha\beta}Tr(\Gamma_{\mu\nu})Tr(V_\alpha - \frac{1}{g}\Gamma_\alpha)Tr(\Delta_\beta) \ ,
\end{aligned} \tag{4.4}$$

where g_i, $(i = 1,...6)$ are arbitrary coefficient functions of the variable X. By rearranging terms we may write $L_{anomalous}$ in a compact form, e.g.,

$$L_{VP\gamma} = g_V\frac{e}{F_\pi}\epsilon^{\mu\nu\alpha\beta}\partial_\mu A_\nu Tr(Q\{\partial_\alpha V_\beta, \mathcal{P}\}) \ , \tag{4.5}$$

$$L_{P\gamma\gamma} = g_P\frac{e^2}{2F_\pi}\epsilon^{\mu\nu\alpha\beta}\partial_\mu A_\nu\partial_\alpha A_\beta Tr(\{Q^2, \mathcal{P}\}) \ . \tag{4.6}$$

Here g_P and g_V represent certain combinations of the function coefficients g_i and constants in L_{WZW}. As in section II, the direct symmetry breaking terms are constructed by inserting the quantity B, Eqn. 2.16, in the expressions above, i.e.,

$$\bar{L}_{VP\gamma} = g_V\frac{e}{F_\pi}\epsilon^{\mu\nu\alpha\beta}\partial_\mu A_\nu Tr(Q\{B, \{\partial_\alpha V_\beta, \mathcal{P}\}\}) \ , \tag{4.7}$$

$$\bar{L}_{P\gamma\gamma} = g_P\frac{e^2}{2F_\pi}\epsilon^{\mu\nu\alpha\beta}\partial_\mu A_\nu\partial_\alpha A_\beta Tr(\{Q^2, \{B, \mathcal{P}\}\}) \ . \tag{4.8}$$

A. The $V \to P\gamma$ and $P \to V\gamma$ Processes

The relevant vertices for a vector (pseudoscalar) meson decaying into a pseudscalar (vector) meson and a photon are,

$$\mathcal{V}(VP\gamma) = -ig_V\frac{e}{F_\pi}v(VP)\epsilon^{\mu\nu\alpha\beta}k_\mu e^{(\gamma)}_\nu p_\alpha e^{(V)}_\beta \ , \tag{4.9}$$

where $e^{(V)}_\nu$ (p) and $e^{(\gamma)}_\nu$ (k) are the polarization (four-momentum) of the vector meson and final photon, respectively. The quantities v incorporate all internal symmetries of the processes under discussion and are listed in Table II. In terms of these vertices the widths of the decays $V \to P\gamma$ and $P \to V\gamma$ are,

$$\Gamma(VP\gamma) = G_V\frac{(m_V^2 - m_P^2)^3}{m_V^3 F_8^2}|v(VP)|^2 \ , \tag{4.10}$$

$$\Gamma(PV\gamma) = 3G_V\frac{(m_P^2 - m_V^2)^3}{m_P^3 F_8^2}|v(VP)|^2 \ , \tag{4.11}$$

with,

$$G_V = \frac{e^2}{4\pi}\frac{g_V^2}{24} \ . \tag{4.12}$$

B. The $P \to \gamma\gamma$ decays

The vertices for decays of a pseudoscalar meson into two photons are,

$$\mathcal{V}(P\gamma\gamma) = -2ig_P \frac{e^2}{F_8} \bar{v}(P) \epsilon^{\mu\nu\alpha\beta} k_{1\mu} e_\nu^{(\gamma)} k_{2\alpha} e_\beta^{(\gamma)} \ , \tag{4.13}$$

with $e_\nu^{(\gamma)}$ and $e_\alpha^{(\gamma)}$ being the polarizations of the final photons, and k_1, k_2 their corresponding four-momenta. Again, the functions $\bar{v}(P)$ contain the internal symmetry information and are listed in Table II. With these vertices the decay rate is given by,

$$\Gamma(P\gamma\gamma) = G_P \frac{m_P^3}{F_8^2} |\bar{v}(P)|^2 \ , \tag{4.14}$$

where,

$$G_P = \frac{\pi}{2} \left(\frac{e^2}{4\pi}\right)^2 \left(\frac{g_P}{9}\right)^2 \ . \tag{4.15}$$

C. Numerical Analysis and Results

The expressions quoted above for the decay widths involve three symmetry breaking scales ($c_A,\ d_A,\ r$), a direct symmetry breaking scale (c_W), and two coupling constants ($g_V,\ g_P$). We fix the coupling constants from experiment and we treat the other quantities as free model parameters to be determined from global fit. As a general rule the experimental masses and decay widths used are the best fit values reported in the latest review of particle properties [37]. The pion radiative constants is taken to be $F_\pi = F_8 = 93MeV$.

Consider now the values of the coupling constants. From the experimental width of the $\omega \to \pi\gamma$ decay, and Eqn. 4.10, one obtains $\Gamma(\omega\pi\gamma) = G_V(m_\omega^2 - m_\pi^2)^3/(m_\omega^3 F_8^2) = (716 \pm 43)KeV$ [37]. This relation yields,

$$G_V = (1.44 \pm 0.04) \cdot 10^{-5} \ , \qquad g_V = 0.22 \pm 0.006 \ . \tag{4.16}$$

In practice, these same constants are obtained from a similar relation, $\Gamma(\rho\pi\gamma) = G(m_\rho^2 - m_\pi^2)^3/(m_\rho^3 F_8^2) = (76 \pm 10)KeV$ [37] for the $\rho \to \pi\gamma$ decay. Similarly, from the decay width of $\pi \to \gamma\gamma$ one finds [37], $\Gamma(\pi^0\gamma\gamma) = 9G_P m_\pi^3/2F_8^2 = (7.8 \pm 0.55)eV$ [37],

$$G_P = (4.9 \pm 0.07) \cdot 10^{-8}, \qquad g_P = 0.073 \pm 0.001 \ . \tag{4.17}$$

Furthermore, the vector meson strange-nonstrange admixture parameter ϵ can be deduced from the decay widths of the ϕ and ω mesons into a pion and a photon. The $\phi \to \pi\gamma$ decay is forbidden in the limit of ideal mixing. However, from Eqns. 4.10 and the experimental decay rates [37] one obtains,

$$\frac{\Gamma(\phi\pi^0\gamma)}{\Gamma(\omega\pi^0\gamma)} = 2\epsilon^2 \left[\frac{(m_\phi^2 - m_\pi^2)m_\omega}{(m_\omega^2 - m_\pi^2)m_\phi}\right]^3 = \frac{(5.8 \pm 0.6)KeV}{(716 \pm 43)KeV} = 0.008 \pm 0.001 \ . \tag{4.18}$$

This ratio gives $\epsilon = 0.0432 \pm 0.004$, a value which is equal within experimental error, to that deduced from vector mass matrix diagonalization(see section II). The calculated decay widths and parameter values as obtained from global fit to data using different mixing schemes are summarized in Tables III and IV.

Based on the χ^2/dof values listed in Tables III and IV, the one mixing angle *Alternative I* scheme provides the best explanation of the data. This observation remains valid, should we have used the older set of data [38], though the resulting fit qualities are slightly poorer.

Alternative I -One mixing angle scheme First we recall that in this case $f = 1$, $\kappa_{08} = 0$ and $\lambda = 0$. This simplifies Eqns.3.5,3.10,3.17, 3.24 and allows expressing the mixing angle in terms of the physical meson masses,

$$\frac{\tan\theta_P}{1 - \tan^2\theta_P} = -\sqrt{\left(\frac{m_{\eta'}^2 - m_\eta^2}{m_{\eta'}^2 + m_\eta^2 - 2\mu_{88}^2 z^2}\right)^2 - 1} \ , \tag{4.19}$$

where $\mu_{88}^2 z^2$ is determined by the pion and kaon mass and rescaling parameters, i.e.,

$$\mu_{88}^2 z^2 = \frac{m_\pi^2}{3}\left(4\frac{m_K^2}{m_\pi^2}\frac{1}{z_K^2} - 1\right) z^2 \ . \tag{4.20}$$

The rescaling parameter z must not be considerably smaller than z_K. Indeed, from Eqn. 3.13 for the Kaon mass and Eqns. 3.21 for the η and η' masses we may write,

$$\frac{4}{3}\frac{m_K^2}{m_\eta^2}\frac{1}{z_K^2} - \frac{1}{3}\frac{m_\pi^2}{m_\eta^2} \geq \frac{1}{z^2} \ , \tag{4.21}$$

or equivalently, $1.03z \geq z_K$. Thus z is at most a factor of $\sim 3\%$ smaller than z_K. With the *Alternative I* parameter set of Table III, the rescaling parameters and mixing angle are,

$$z_K = 0.87 \pm 0.04, \quad z = 0.89 \pm 0.06, \quad \theta_P = -(14.3 \pm 2.2)^o \ . \tag{4.22}$$

How significant are the departure of these parameters from their values at the exact limit of $U(3)_L \bigotimes U(3)_R$ symmetry? Clearly, the exact $SU(3)_F$ symmetry limit values $c_W = 0$, $c_A = d_A = 0$, $r = 1$ would be inconsistent with the measured value of the ratio $\Gamma(K^{*0}K^0\gamma)/\Gamma(K^{*+}K^+\gamma)$; with $c_W = 0$ this ratio becomes 4 as opposed to the experimental value of 2.34 ± 0.43. In addition, direct symmetry breaking alone is not sufficient; with $c_A = 0$ and $c_W = -0.2$, the calculated width $\Gamma(K^{*0}K^0\gamma)$ is about 30% higher than experimental value, well beyond the measurement accuracy. Based on fit quality both direct (i.e. $c_W \neq 0$) and indirect (i.e. $c_A \neq 0$) symmetry breaking terms are needed to explain data.

As a further check we may estimate these scales using data directly. First, from the ratio,

$$\frac{\Gamma(K^{*0}K^0\gamma)}{\Gamma(K^{*+}K^+\gamma)} = 4\left[\frac{1+\frac{1}{2}c_W}{1-c_W}\right]^2 = \frac{(117 \pm 10)KeV}{(50 \pm 5)KeV} = 2.34 \pm 0.43 \ , \tag{4.23}$$

one obtains $c_W = -0.19 \pm 0.04$ in full agreement with the value of the *Alternative I* parameter set. Next, the decay width $\Gamma(K^{*0}K^0\gamma)$ involves the kaon rescaling parameter (or equivalently the scales c_A). One finds, $z_K = 0.86 \pm 0.08$ and $c_A = 0.52 \pm 0.22$. The other two parameters are deduced from the ratios, $\Gamma(\phi\eta\gamma)/\Gamma(\omega\eta\gamma)$ and, $\Gamma(\phi\eta\gamma)/\Gamma(\eta'\rho\gamma)$, which yield,

$$d_A = -0.45 \pm 0.2 \ , \quad r = 0.98 \pm 0.1 \ . \tag{4.24}$$

With this set of parameters the corresponding mixing angle $\theta_P = -(16.2 \pm 2.4)^o$ and $\chi^2/dof = 36/6$ is more than a factor of 10 higher as compared to the *Alternative I* global fit results. A global fit to data with the assumption of an exact nonet symmetry, i.e. $r = 1$, gives $c_W = -0.18 \pm 0.05$, $c_A = 0.64 \pm 0.06$, $d_A = -0.32 \pm 0.04$ and $\theta_P = -(15.2 \pm 2.2)^o$ but with $\chi^2/dof = 13.2/7$. Again the fit quality is significantly poorer as compared to the *Alternative I* results.

It is worthy of mention that the mixing angle is very sensitive to the ratio of the rescaling parameters z and z_K. The smallest mixing angle is obtained with $z = z_K$ (i.e., $d_A = -c_A/4$) and readily grows for increasing z/z_K. However, a global fit with these parameters taken to be equal is again far poorer with $\chi^2/dof = 8.2/7$ and $c_W = -0.26 \pm 0.05$, $c_A = 0.56 \pm 0.06$, $r = 0.86 \pm 0.1$ $\theta_P = -(8.2 \pm 2.0)^o$. The mixing angle is far less sensitive to the direct symmetry breaking scale c_W. Based on these global fit analyses we may conclude that nonet symmetry breaking and field rescaling parameters depart slightly but *significantly* from unity.

Finally, by comparing Eqn. 4.20, with the familiar quark model result [37],

$$m_{88}^2 = \frac{m_\pi^2}{3}\left(4\frac{m_K^2}{m_\pi^2} - 1)\right)(1+\Delta) \ , \tag{4.25}$$

the symmetry breaking measure is,

$$\Delta \approx \frac{z^2}{z_K^2} - 1 \ . \tag{4.26}$$

The *Alternative I* parameters correspond to $\Delta = 0.05 \pm 0.01$.

The *Alternative II* -Two mixing angle scheme The mixing angles and rescaling parameters corresponding to the *Alternative II* fit parameters of Table III are :
$\lambda = (25.1 \pm 2)^o$, $\chi = -(34.9 \pm 3)^o$, $z_K = 0.95 \pm 0.06$, $z = 1.0 \pm 0.06$, $f = 0.88 \pm 0.06$, or equivalently,

$$\theta_\eta = -(5.8 \pm 2)^o, \quad \theta_{\eta'} = -(12.7 \pm 2)^o, \quad z_1 = 0.98 \pm 0.05, \quad z_2 = 0.96 \pm 0.05 \ . \tag{4.27}$$

In view of the very poor fit quality (confidence level of less than 0.05), it seems quite unjustified to use a two mixing angle scheme in analyzing radiative decays.

Quark flavor basis scheme As indicated, this scheme corresponds to the *Alternative II* in the limit of exact nonet symmetry ($r = 1$). In this case the values of c_A and d_A are considerably different, and the rescaling parameters and mixing angle are,

$$\phi = (40.0 \pm 2.0)^o, \quad z_K = 0.76 \pm 0.04, \quad z_s = 0.88 \pm 0.05 \ . \tag{4.28}$$

With the standard octet-singlet mixing angle defined as $\theta_P = \phi - \theta_{ideal}$ as in Ref. [10] we have $\theta_P = -(14.7 \pm 2.0)^o$. This value agrees remarkably well with the *Alternative I* results. Yet, with $\chi^2/dof = 19.8/7$ (confidence level less than 0.01) this scheme like the *Alternative II* is far poorer than the *Alternative I*.

It is of interest to note that similar mixing angle values were extracted by several authors using various phenomenological models. Here we mention few examples. Feldmann et al. [10,11] using their QFB analysis reported a value $\theta_P = -(14.8 \pm 2.9)^o$. Cao and Signal [20] obtained value of $\theta_P = -(14.5 \pm 2.0)^o$ from analyzing large momentum $e^+e^- \to \eta, \eta' \to 2\gamma$. Somewhat a less negative value, $\theta_P = -(11.59 \pm 0.76)^o$, was determined by Benayoun et al. [15,16]. The fact that different model analyses predict similar results should not be surprising since the mixing angle is sensitive to ratios of scaling parameters (see Eqn. 4.19) rather than to their actual values. In fact, since this ratio is close to one, the mixing angle is fixed rather accurately by experimental meson masses. This also means that in similar analyses where rescaling is neglected or the corresponding ratios of parameters are close to unity, as in our *Alternative I* case, similar values of θ_P are expected.However as our analyses show rescaling plays an important role in explaining the decay widths.

V. SUMMARY AND DISCUSSION

In this paper, using a slightly generalized version of the HLS approach of Bando et al. [26] and a general procedure of including the η' into a chiral theory, we have constructed an effective lagrangian which incorporates "indirect" as well as "direct" symmetry breaking effects. At lowest order the lagrangian comprises two terms L_A, L_V describing respectively, the interactions of pseudoscalar and vector meson nonets, and a "kinetic" term for the vector mesons. In a way similar to that proposed by Bramon et al. [5] and Benayoun et al. [13,15]. we have constructed "direct" asymmetric companions $\bar{L}_A$ and $\bar{L}_V$ using a matrix B which is proportional to quark mass matrix. With this choice of B, our theory predicts the same ratios of isospin to $SU(3)$ symmetry breaking scales as in QCD. The kinetic and mass lagrangian terms are non-diagonal but can be reduced into the standard quadratic form by transforming the intrinsic fields into the orthogonal physical ones via rescaling and two unitary transformations. Most importantly, this procedure determines the particle mixing in a unique way [31].

In the numerical analyses We have used different particle mixing schemes : (i) the *Alternative I*, (ii) the *Alternative II* and (iii) the QFB scheme. The results obtained with these possibilities reflect upon the nature of the symmetry breaking required to explain radiative decay widths. Based on global fit analyses, we observe that the *Alternative I* provides by a far better description of the data considered. This argues for $SU(3)_F$ symmetry breaking supplemented with broken nonet symmetry. The other alternatives of $U(3)_F$ symmetry breaking with or without nonet symmetry breaking yield rather poor fits and seem unjustified.

There have been numerous data analyses in the last three decades attempting to deduce a reliable and accurate value of the pseudoscalar mixing angle [1,2,6–8,13,17,15,20]. The values reported range from $\theta = -23^o$ to as high as $\theta = -10^o$. In marked difference with these previous studies, in the present work the particle mixing schemes are well related to the lagrangians presumed. Albeit, we believe that the theory proposed furnishes an accurate framework for the study of electroweak and strong interactions amongst light flavor mesons.

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	Alternative I	*Alternative II*	QFB
X_η	$z\cos\theta_P - \sqrt{2}r\sin\theta_P$	$z\cos\lambda\cos\chi - f\sin\lambda\sin\chi +$ $\sqrt{2}r(-z\sin\lambda\cos\chi - f\cos\lambda\sin\chi)$,	$\sqrt{3}\cos\phi$,
Y_η	$-2z\cos\theta_P - \sqrt{2}r\sin\theta_P$	$-2(z\cos\lambda\cos\chi - f\sin\lambda\sin\chi)+$ $\sqrt{2}r(-z\sin\lambda\cos\chi - f\cos\lambda\sin\chi)$,	$-\sqrt{6/\kappa_s}\sin\phi$,
X'_η	$z\sin\theta_P + \sqrt{2}r\cos\theta_P$	$z\cos\lambda\sin\chi + f\sin\lambda\cos\chi +$ $\sqrt{2}r(-z\sin\lambda\sin\chi + f\cos\lambda\cos\chi)$,	$\sqrt{3}\sin\phi$
Y'_η	$-2z\sin\theta_P + \sqrt{2}r\cos\theta_P$	$-2(z\cos\lambda\sin\chi + f\sin\lambda\cos\chi)+$ $\sqrt{2}r(-z\sin\lambda\sin\chi + f\cos\lambda\cos\chi)$.	$\sqrt{6/\kappa_s}\cos\phi$.

TABLE I. The X_i and Y_i coefficients, Eqns. 3.26. Mixing angles are θ_P for the *Alternative I* and ϕ for the QFB scheme.

$v(\rho\pi) = \frac{1}{3}$	$v(\rho\eta) = \frac{1}{\sqrt{3}} X_\eta$	$v(\rho\eta') = \frac{1}{\sqrt{3}} X_{\eta'}$
$v(\omega\pi) = 1$	$v(\omega\eta) = -\frac{1}{3\sqrt{3}} X_\eta$	$v(\omega\eta') = \frac{1}{3\sqrt{3}} X_{\eta'}$
$v(\phi\pi) = \sqrt{2}\epsilon,$	$v(\phi\eta) = -\frac{\sqrt{2}}{3\sqrt{3}} Y_\eta (1 + c_W)$	$v(\phi\eta') = -\frac{\sqrt{2}}{3\sqrt{3}} Y_{\eta'} (1 + c_W)$
$v(K^{*0}K^0) = v(\bar{K}^{*0}\bar{K}^0) = -\frac{2}{3} z_K (1 + \frac{1}{2} c_W)$		
$v(K^{*+}K^+) = v(K^{*-}K^-) = \frac{1}{3} z_K (1 - c_W)$		
$\bar{v}(\pi) = \frac{3}{\sqrt{2}}$	$\bar{v}(\eta) = \frac{1}{\sqrt{6}} [5X_\eta + (1 + 2c_W) Y_\eta]$	$\bar{v}(\eta') = \frac{1}{\sqrt{6}} [5X_{\eta'} + (1 + 2c_W) Y_{\eta'}]$

TABLE II. The internal symmetry factors $v(VP)$ and $v(\bar{P})$, Eqns. 4.9, 4.13.

	c_W	c_A	d_A	r	χ^2/dof
Alternative I	$-(0.20 \pm 0.05)$	(0.64 ± 0.06)	-0.25 ± 0.04	0.91 ± 0.04	$3.1/6$
Alternative II	$-(0.27 \pm 0.05)$	0.2 ± 0.05	0.1 ± 0.02	0.94 ± 0.05	$18.6/6$
QFB	$-(0.19 \pm 0.05)$	(1.4 ± 0.1)	-1.1 ± 0.1	*1	$19.8/7$

TABLE III. Symmetry breaking scales and χ^2/dof from global fit to data. Values marked with an asterisk were kept fixed.

Decay	Γ_{exp}(KeV)	Γ_{calc}(KeV)		
		Alternative I	*Alternative II*	QFB
$\rho \to \pi\gamma$	76 ± 10	76 ± 10	76 ± 10	76 ± 10
$\omega \to \pi\gamma$	716 ± 43	* 716	* 716	* 716
$\rho \to \eta\gamma$	36 ± 12	40.0 ± 4.1	47.4 ± 4.2	52.8 ± 4.5
$\omega \to \eta\gamma$	5.5 ± 0.85	5.2 ± 0.45	6.1 ± 0.5	6.8 ± 0.6
$\phi \to \eta\gamma$	57.8 ± 1.6	60.8 ± 2.4	65.7 ± 3.5	60 ± 3
$\phi \to \eta'\gamma$	$0.30 \begin{array}{l} +0.20 \\ -0.16 \end{array}$	0.37 ± 0.03	0.24 ± 0.03	0.4 ± 0.05
$\eta' \to \rho\gamma$	61.2 ± 7.5	70.7 ± 5.9	70.3 ± 4.1	78.5 ± 6.6
$\eta' \to \omega\gamma$	6.12 ± 0.75	6.47 ± 0.45	6.4 ± 0.4	7.2 ± 0.5
$\pi^0 \to \gamma\gamma$	$(7.8 \pm 0.55)10^{-3}$	* $7.8 \cdot 10^{-3}$	* $7.8 \cdot 10^{-3}$	* $7.8 \cdot 10^{-3}$
$\eta \to \gamma\gamma$	0.460 ± 0.050	0.468 ± 0.03	0.58 ± 0.02	0.637 ± 0.03
$\eta' \to \gamma\gamma$	4.280 ± 0.280	4.02 ± 0.3	3.68 ± 0.24	4.52 ± 0.3
$K^{*0} \to K^0\gamma$	117 ± 10	106 ± 8.3	120.7 ± 4.6	113 ± 8
$K^{*\pm} \to K^{\pm}\gamma$	50 ± 5	48.6 ± 3.2	63.3 ± 5.2	48 ± 5
χ^2/dof		3.1/6	18.8/6	19.8/7

TABLE IV. Calculated radiative decay widths. The *Alternative I*, *Alternative II* and QFB correspond to the parameter sets of Table III. The widths marked with an asterisk were used to evaluate the coupling constants g_P and g_V (see text). The experimental decay widths of the second column are the best fit values of Ref. [37].

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